Example 2: calculate f(0,1,1), f(1,0,0), f(1,1,1)

$$f(a,b,c) = (a+b+c)\,(a + \bar{b}\,c) + c\,\overline{(a+c)}$$

NOT

x	z
0	1
1	0

$z = \bar{x}$

AND

x y	z
0 0	0
0 1	0
1 0	0
1 1	1

$z = x \cdot y$

OR

x y	z
0 0	0
0 1	1
1 0	1
1 1	1

$z = x + y$

$$f(0,1,1) = (0+1+1)(0+\bar{1}\,1) + 1\,\overline{(0+1)}$$

$$= (1+1)(0+0\cdot 1) + 1\,\bar{1}$$

$$= 1\cdot(0+0) + 1\cdot 0 = 1\cdot 0 + 0 = 0+0 = \underline{\underline{0}}$$

$$f(1,0,0) = (1+0+0)(1+\underset{1}{\bar{0}}\,\underset{0}{0}) + 0\,\overline{(1+0)}$$

$$= 1\cdot 1 + 0 = \underline{\underline{1}}$$

$$f(1,1,1) = (1+1+1)(1+\bar{1}\,1) + 1\,\overline{(1+1)}$$

$$= 1\cdot 1 + 0 = \underline{\underline{1}}$$

Example 3. Convert to normalized forms of SOP and POS

$$(a+b+c)\,(ab + \bar{b}c) + cd\,\overline{(a+c)} =$$

Identity	$x+0 = x$	$x \cdot 1 = x$
Commutativity	$x+y = y+x$	$x\cdot y = y\cdot x$
Distributivity	$x\cdot(y+z) = (x\cdot y)+(x\cdot z)$	$x+(y\cdot z) = (x+y)\cdot(x+z)$
Complement	$x+\bar{x} = 1$	$x\cdot\bar{x} = 0$

Idempotence	$x+x = x$	$x\cdot x = x$
Complement uniqueness	$\bar{x}$ is unique	
Annihilation	$x+1 = 1$	$x\cdot 0 = 0$
Double complement	$\overline{\overline{(x)}} = x$	
Absorption	$x+xy = x$	$x\cdot(x+y) = x$
Consensus	$x+\bar{x}y = x+y$	$x\cdot(\bar{x}+y) = x\cdot y$
Associativity	$x+(y+z) = (x+y)+z$	$x\cdot(y\cdot z) = (x\cdot y)\cdot z$
De Morgan	$\overline{x\cdot y} = \bar{x}+\bar{y}$	$\overline{x+y} = \bar{x}\cdot\bar{y}$
Reduction	$xy+x\bar{y} = x$	$(x+y)(x+\bar{y}) = x$

$$= (a+b+c)ab + (a+b+c)\bar{b}c +$$

$$+ cd\,\bar{a}\,\bar{c} = aab + abb + abc$$

$$+ \cancel{ac\bar{c}d}^{\to 0} = \underbrace{ab + abc}_{SOP} = ab$$

$$(a+b+c)\,(ab + \bar{b}c) + cd\,\overline{(a+c)} = (a+b+c)(ab+\bar{b})(ab+c) + \cancel{cd\bar{a}\bar{c}}^{\to 0} =$$

$$= (a+b+c)(a+\bar{b})(\cancel{b+\bar{b}})^{1}(a+c)(b+c) =$$

$$= \cancel{(a+b+c)}^{\to 0}(a+\bar{b})(a+c)(b+c)$$

POS

Example 4. Convert from normalized to canonical forms.

Identity	x+0 = x	x · 1 = x
Commutativity	x+y = y+x	x·y = y·x
Distributivity	x·(y+z) = (x·y)+(x·z)	x+(y·z) = (x+y)·(x+z)
Complement	$x+\bar{x} = 1$	$x\cdot\bar{x} = 0$

$$a + a\bar{b}c + \bar{b}c =$$

b,c   a

$$a(\underbrace{b+\bar{b}}_{1})(\underbrace{c+\bar{c}}_{1}) + a\bar{b}c + (\underbrace{a+\bar{a}}_{1})\bar{b}c$$

$$\underbrace{x}+\underbrace{x} = x$$

$$=(ab+a\bar{b})(c+\bar{c}) + a\bar{b}c + a\bar{b}c + \bar{a}\bar{b}c$$

$$= abc + a\bar{b}c + ab\bar{c} + a\bar{b}\bar{c} + \cancel{a\bar{b}c} + \cancel{a\bar{b}c} + \bar{a}\bar{b}c =$$

$$= \underset{111\;7}{abc} + \underset{101\;5}{a\bar{b}c} + \underset{6}{ab\bar{c}} + \underset{4}{a\bar{b}\bar{c}} + \underset{1}{\bar{a}\bar{b}c} = \Sigma(1,4,5,6,7)$$

(a,b,c)

$$(a+\bar{b}+c)(a+b)(a+\bar{c}) = (a+\bar{b}+c)(a+b+\underbrace{c\bar{c}}_{\phi})(a+b\bar{b}+c) =$$

c   b

$$x \cdot x = x$$

$$(a+\bar{b}+c)(a+b+c)(a+b+\bar{c})\cancel{(a+b+c)}\cancel{(a+\bar{b}+c)}$$

$$= \underset{010\;\;2}{(a+\bar{b}+c)}\underset{000\;\;0}{(a+b+c)}\underset{001\;\;1}{(a+b+\bar{c})} = \Pi(0,1,2)$$

(a,b,c)

Example 5. Convert from SOP/POS to truth table

$z(a,b,c) = a\bar{c} + a\,\bar{b}\,c + \bar{a}c + \bar{b}c$

1

$z = 1$ iif $a\bar{c} = 1 : a=1, c=0$

or

$a\bar{b}c = 1 : a=1, b=0, c=1$

or

$a=0, c=1$

or

$b=0, c=1$

a b c	z
0 0 0	0
→ 0 0 1	1
0 1 0	0
→ 0 1 1	1
→ 1 0 0	1
→ 1 0 1	1
→ 1 1 0	1
1 1 1	0

$z(a,b,c) = (a+b+c)(a+\bar{b})(a+c)$

$z = 0$, iif $(a+b+c)=0 : a=0, b=0, c=0$

or

$a=0, b=1$

or

$a=0, c=0$

a b c	z
→ → 0 0 0	0
0 0 1	1
→ → 0 1 0	0
→ 0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

Example 6. Algebraic logic expression minimization (Quine-McCluskey simplified method)

$x y + \bar{x} y = y$

$$F(a,b,c,d) = \Sigma(0,1,4,9,11,13,15)$$

$$F(a,b,c,d) = \bar{a}\,\bar{b}\,\bar{c}\,\bar{d} + \bar{a}\,\bar{b}\,\bar{c}\,d + \bar{a}\,b\,\bar{c}\,\bar{d} + a\,\bar{b}\,\bar{c}\,d + a\,\bar{b}\,c\,d + a\,b\,\bar{c}\,d + a\,b\,c\,d$$

		1st order imp.	2nd order imp.
→ 0	$\cancel{\bar{a}\,\bar{b}\,\bar{c}\,\bar{d}}$	→ $\bar{a}\bar{b}\bar{c}\ (0,1)$ ✓	$ad\ (9,11,13,15)$ *
→ 1	$\cancel{\bar{a}\,\bar{b}\,\bar{c}\,d}$	→ $\bar{a}\bar{c}\bar{d}\ (0,4)$ *	
→ 4	$\cancel{\bar{a}\,b\,\bar{c}\,\bar{d}}$	→ $\bar{b}\bar{c}d\ (1,9)$ ✓	
→ 9	$\cancel{a\,\bar{b}\,\bar{c}\,d}$	→ $\cancel{a\bar{b}d}\ (9,11)$	
→ 11	$\cancel{a\,\bar{b}\,c\,d}$	→ $\cancel{a\bar{c}d}\ (9,13)$	
→ 13	$\cancel{a\,b\,\bar{c}\,d}$	→ $\cancel{acd}\ (11,15)$	
→ 15	$\cancel{a\,b\,c\,d}$	→ $\cancel{abd}\ (13,15)$	

(*) Essential prime implicant (we have to use these in our expression)

(✓) We need to use one of these also to cover all the minterms.

Two solutions with the same complexity:

1) $F = \bar{a}\bar{c}\bar{d} + ad + \bar{b}\bar{c}d$

2) $F = \bar{a}\bar{c}\bar{d} + ad + \bar{a}\bar{b}\bar{c}$

} minimal SOP

Example 7. Logic expression minimization using K-maps.

$F(a,b,c,d) = \Sigma(0,1,4,9,11,13,15)$

cd \ ab	00	01	11	10
00	1	1		
01	1		1	1
11			1	1
10				

F

$\bar{a}\,\bar{c}\,\bar{d}$

0 0 0

$\bar{b}\,\bar{c}\,d$

ad

$F = \bar{a}\bar{c}\bar{d} + \bar{b}\bar{c}d + ad$ minimal SOP

$F(a,b,c,d) = \Sigma(0,1,4,9,11,13,15)$

P. S. O.

maxterms

cd \ ab	00	01	11	10
00	1	1	0	0
01	1	0	1	1
11	0	0	1	1
10	0	0	0	0

F

$(\bar{a} + d)$

1 0

$0 + 0 = 0$

$(a + \bar{b} + \bar{d})$

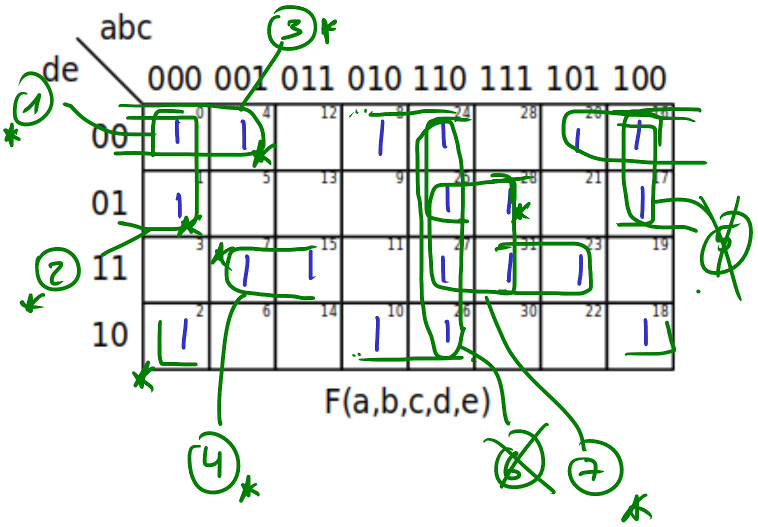
$(a + \bar{c})$

$F = (\bar{a}+d)(a+\bar{b}+\bar{d})(a+\bar{c})$ minimal POS.

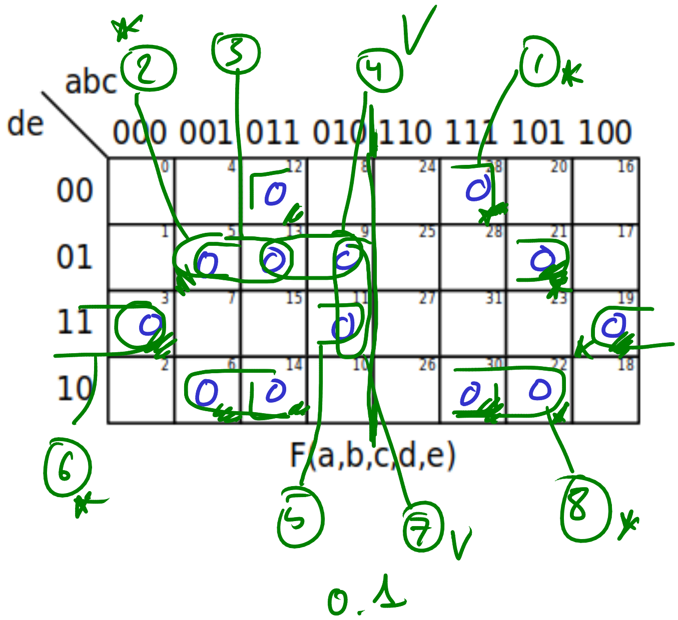
Extra example: 5 variables K-map minimization.

F(a,b,c,d,e) = Σ(0,1,2,4,7,8,10,15,16,17,18,20,23,24,25,26,27,28,31)

SOP



POS



$F = \bar{c}\bar{e} + \bar{b}\bar{c}\bar{d} + \bar{b}\bar{d}\bar{e} + cde + abe$

(1) (2) (3) (4) (7)

minimal SOP

$F = (\bar{b}+\bar{c}+e)(b+\bar{c}+d+\bar{e})(b+c+\bar{d}+\bar{e})(a+\bar{b}+d+\bar{e})\cdot$

(1) (2) (6) (4)

$(a+\bar{b}+\bar{c}+\bar{e})(\bar{c}+\bar{d}+e)$

(7) (8)

minimal POS

Example 8. K-map minimization from different representations

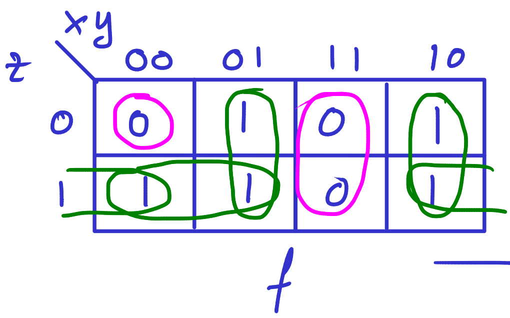
From a list of minterms/maxterms.

$f(\cancel{a,b,c}) = \sum(1,2,3,4,5)$

$f(x,y,z)$

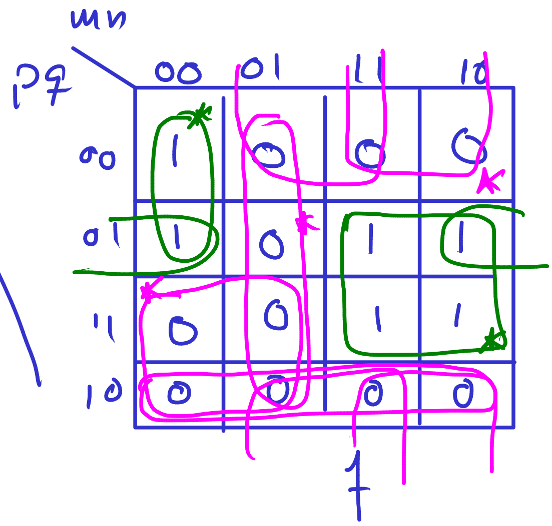
$f(\cancel{a,b,c,d}) = \prod(2,3,4,5,6,7,8,10,12,14)$

$f(m,n,p,q)$



$f = \bar{x}z + \bar{y}z + \bar{x}y + x\bar{y}$ SoP

$f = (x+y+z)(\bar{x}+\bar{y})$ PoS



$f = \bar{m}\bar{n}\bar{p} + mq$

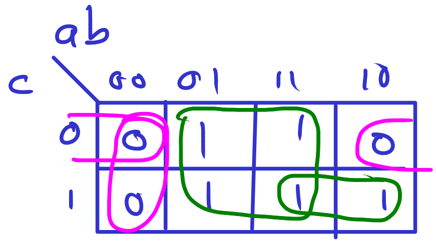
$f = (m+\bar{p})(m+\bar{n})(\bar{m}+q)$

From truth table

	a b c	z
0	0 0 0	0
1	0 0 1	0
2	0 1 0	1
3	0 1 1	1
4	1 0 0	0
5	1 0 1	1
6	1 1 0	1
7	1 1 1	1

$z = \sum(2,3,5,6,7)$

$z = \prod(0,1,4)$



$f = b + ac$

$f = (a+b)(b+c)$

From any expression

$$f(a,b,c,d) = \overline{(a+\bar{b})}(ac+d) + \overline{(b+c)}\,\overline{(\bar{b}+d)}$$

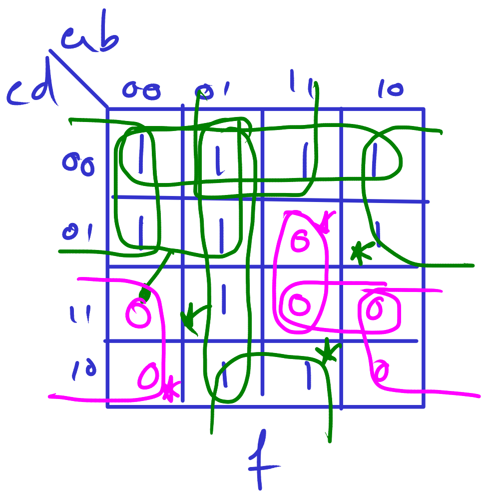
SoP → K-map.

PoS → K-map.

$f = \bar{a}b(ac+d) + \overline{b+c} + \overline{\bar{b}+d} =$

$= \bar{a}bd + \bar{b}\bar{c} + b\bar{d}$

0 1 1   0 0   1 0



$f = \bar{a}\bar{c} + \bar{b}\bar{c} + b\bar{d}$

$f = (\bar{a}+\bar{b}+\bar{d})(b+\bar{c})$

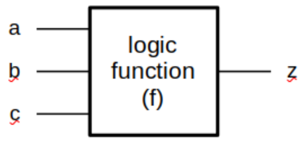
Example 9

Design an optimum two-levels combinational circuit for example 1 (introduction).

Verbal description

A digital alarm system may be on or off and has a presence sensor and a contact sensor at the main door. When the system is on, the alarm will be activated if presence or a door open is detected. When the system is off the alarm is activated only when presence is detected and the door is open (to prevent leaving the door open when at home).

Formal description



$z = f(a,b,c)$

a (on/off switch): 0-on, 1-off
b (presence sensor): 0-no presence, 1-presence.
c (door sensor): 0-door closed, 1-door open
z (alarm): 0-no activated, 1-activated

a b c	z
0 0 0	0
0 0 1	0
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	1
1 1 1	1

c \ ab	00	01	11	10
0	0	0	1	0
1	0	1	1	1

$f = ab + bc + ac$

SOP → AND – OR

cost = 3 + 6 + 0 = 9

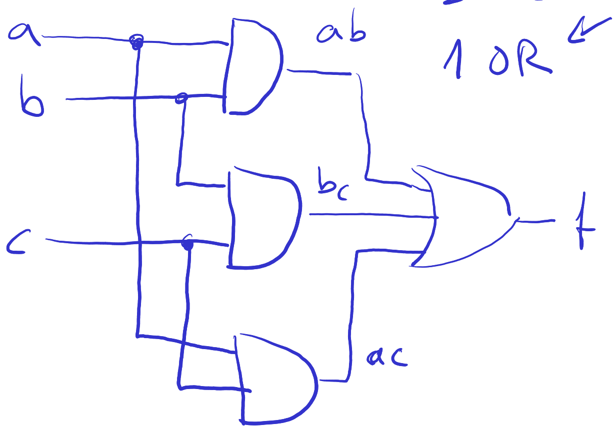
c \ ab	00	01	11	10
0	0	0	1	0
1	0	1	1	1

$f = (a+b)(b+c)(a+c)$

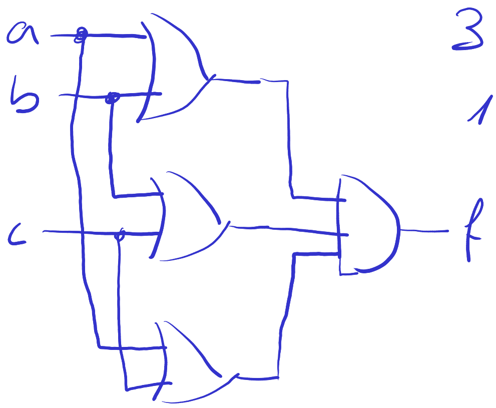
POS → OR – AND

cost = 3 + 6 + 0 = 9

3 AND
1 OR



3 OR
1 AND



Example 10

A modern processor run four processing units at a time: P1, P2, P3 and P4. Each unit sets an output bit 'pi' to one when it is busy. The system is considered busy when any of the following conditions is met:

- P1 and any other unit are busy.
- P2 and P3 are busy.
- P4 is busy and neither P1 nor P2 are busy.

Design a minimum two-level circuit (plus inverters). Inputs are single rail.

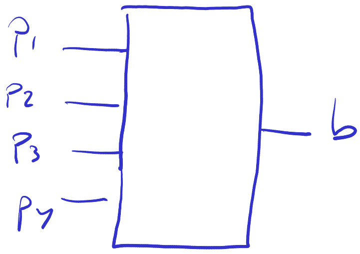
Truth table → K-map

b : busy → b = 1
not busy → b = 0

P_i : 1 – busy
0 – not busy.

P_3P_4 \ P_1P_2	00	01	11	10
00			1	
01	1		1	1
11	1	1	1	1
10		1	1	1

b



$$b = \bar{P_2} P_4 + P_2 P_3 + P_1 P_3 + P_1 P_2$$

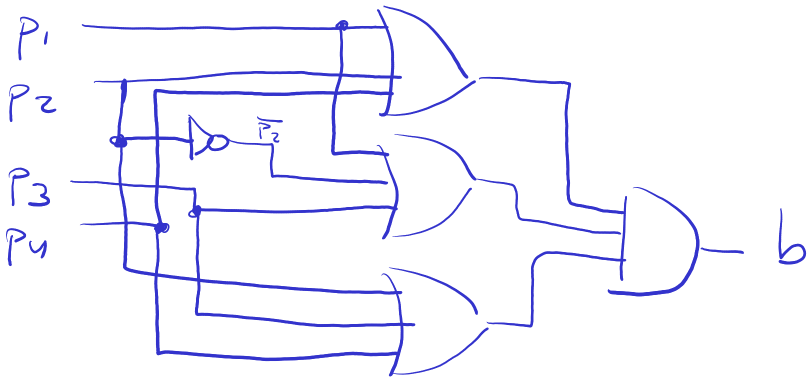
$$cost = 4 + 8 + 1 = 13$$

P_3P_4 \ P_1P_2	00	01	11	10
00	0	0	1	0
01	1	0	1	1
11	1	1	1	1
10	0	1	1	1

b

$$b = (P_1 + P_2 + P_4)(P_1 + \bar{P_2} + P_3)(P_2 + P_3 + P_4)$$

$$cost = 3 + 9 + 1 = 13$$



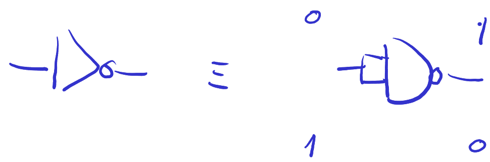
Example 11

A modern processor run four processing units at a time: P1, P2, P3 and P4. Each unit sets an output bit 'pi' to one when it is busy. The system is considered busy when any of the following conditions is met:

- P1 and any other unit are busy.
- P2 and P3 are busy.
- P4 is busy and neither P1 nor P2 are busy.

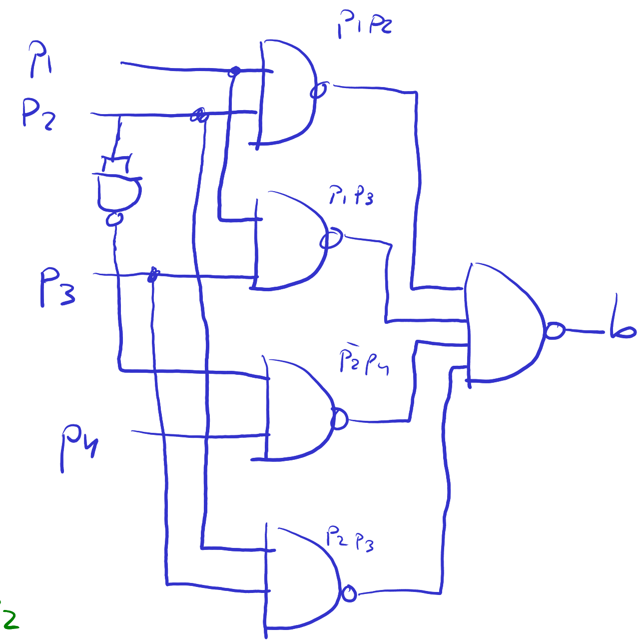
Design a minimum two-levels circuit using only NAND gates. and inverters.

SoP: AND-OR | NAND-NAND



P_3P_4 \ P_1P_2	00	01	11	10
00			1	
01	1		1	1
11	1	1	1	1
10		1	1	1

b



$$b = \bar{P_2} P_4 + P_2 P_3 + P_1 P_3 + P_1 P_2$$

Example 12

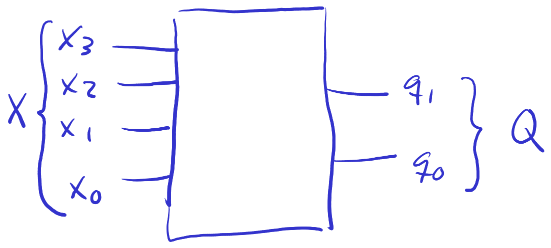
Design a combinational circuit with four inputs (x3, x2, x1, x0) that represent the bits of a BCD digit X, and two outputs (q1, q0) that represents the bits of a magnitude Q, where q is the quotient of the division X/3.

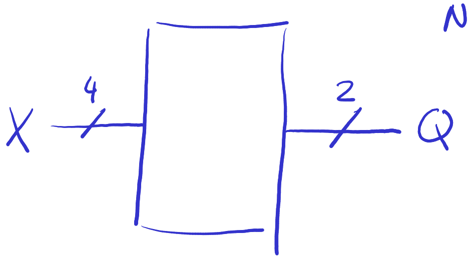
$\rightarrow$ 0 . . . 9

E.g. if X=7 → Q=2, that is, (x3,x2,x1,x0)=(0,1,1,1) → (c1,c0)=(1,0)

0111

Design the circuit using a minimum two-level structure of only NAND gates.





NAND – NAND
in
AND – OR
|||
SoP

BCD

X	$x_3x_2x_1x_0$	q_1q_0
0	0 0 0 0	0 0
1	0 0 0 1	0 0
2	0 0 1 0	0 0
3	0 0 1 1	0 1
4	0 1 0 0	0 1
⋮	⋮	
9	1 0 0 1	1 1
10	1 0 1 0	– –
11	1 0 1 1	– –
⋮	⋮	⋮
15	1 1 1 1	– –

Writing the T.T. is not really needed $\longrightarrow$

x_1x_0 \ x_3x_2	00	01	11	10
00	0 0 [0]	0 1 [4]	– – [12]	1 0 [8]
01	0 0 [1]	0 1 [5]	– – [13]	1 1 [9]
11	0 1 [3]	1 0 [7]	– – [15]	– – [11]
10	0 0 [2]	1 0 [6]	– – [14]	– – [10]

$q_1 \; q_0$

x_1x_0 \ x_3x_2	00	01	11	10
00	0	0	–	1
01	0	0	–	1
11	0	1	–	–
10	0	1	–	–

q_1

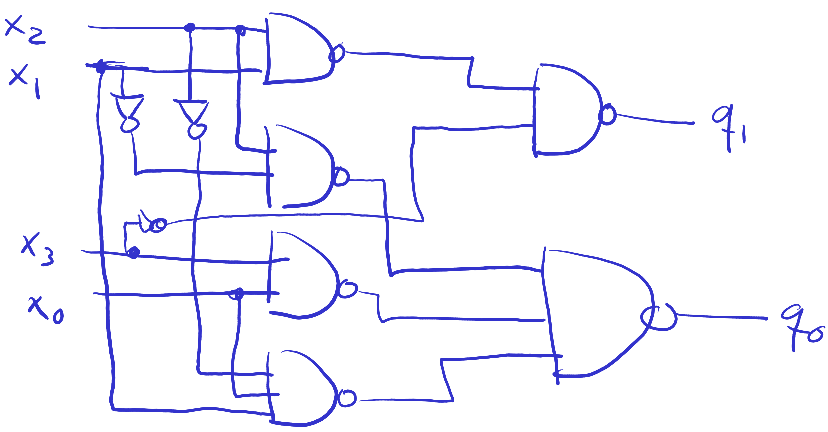
x_1x_0 \ x_3x_2	00	01	11	10
00	0	1	–	0
01	0	1	–	1
11	1	0	–	–
10	0	0	–	–

q_0

$$q_1 = x_3 + x_2x_1$$

$$q_0 = x_2\bar{x}_1 + x_3x_0 + \bar{x}_2x_1x_0$$

$$q_1 = \overline{\overline{x_3 + x_2x_1}} = \overline{(\bar{x}_3)(\overline{x_2x_1})}$$



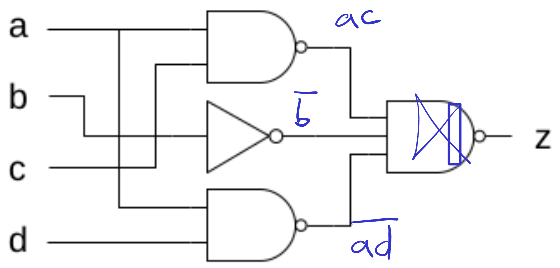
Example 13

The circuit below corresponds to a damaged alarm system with four inputs and one output. Looking at the connections we know that the inputs correspond to:

- a: system activation (0 - off, 1 - on)
- b: fire sensor (0 - no fire, 1 - fire)
- c: front door sensor (0 - close, 1 - open)
- d: presence sensor (0 - no presence, 1 - presence)

When output z is active (z=1) the alarm rings.

- Analyze the circuit and obtain its truth table.
- Describe with words the operation of the alarm: cases that make the alarm to ring, etc.
- Redesign the circuit using only NOR gates.



Circuit ⟶ expression ⟶ K-map ⟶ Truth table

↓

Verbal description

$$z = \overline{\overline{ac}\;\overline{b}\;\overline{ad}} = \overline{\overline{ac}} + \overline{\overline{b}} + \overline{\overline{ad}} =$$

$$= ac + b + ad \quad \text{SoP}$$

NOR-NOR ≡ OR-AND → POS.

abcd	z
0000	0
0001	0
0010	0
0011	0
0100	1
0101	1
0110	1
0111	1
1000	0
1001	1
1010	1
1011	1
1100	1
1101	1
1110	1
1111	1

alarm off (rows 0000–0111)

alarm on (rows 1000–1111)

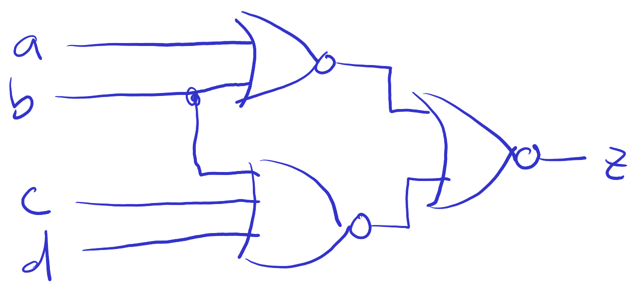
If alarm is off (a=0)
→ Only activates if fire (b=1)
If alarm is on (a=1)
→ Ring in any sensor activation
(fire, presence, door)

cd \ ab	00	01	11	10
00	0	1	1	0
01	0	1	1	1
11	0	1	1	1
10	0	1	1	1

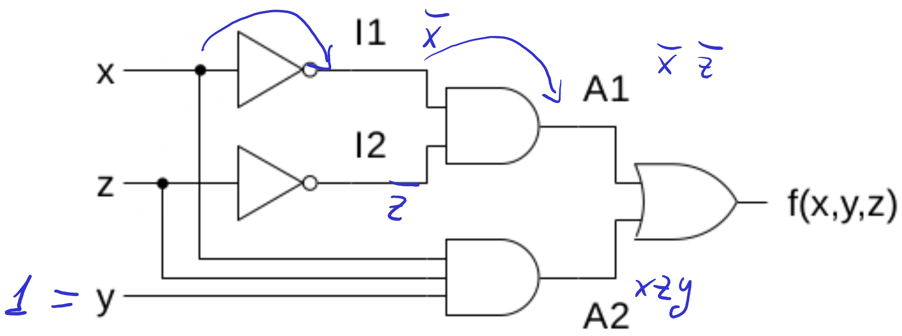
z

$$z = \overline{\overline{(a+b)(b+c+d)}}$$

$$= \overline{\overline{(a+b)} + \overline{(b+c+d)}}$$



Example 14. Obtain the chronogram of output "f" considering that all gates have the same delay "Δ". We know "x" and "z" waveforms and "y=1".



$y = 1$

$I_1 = \bar{x}$

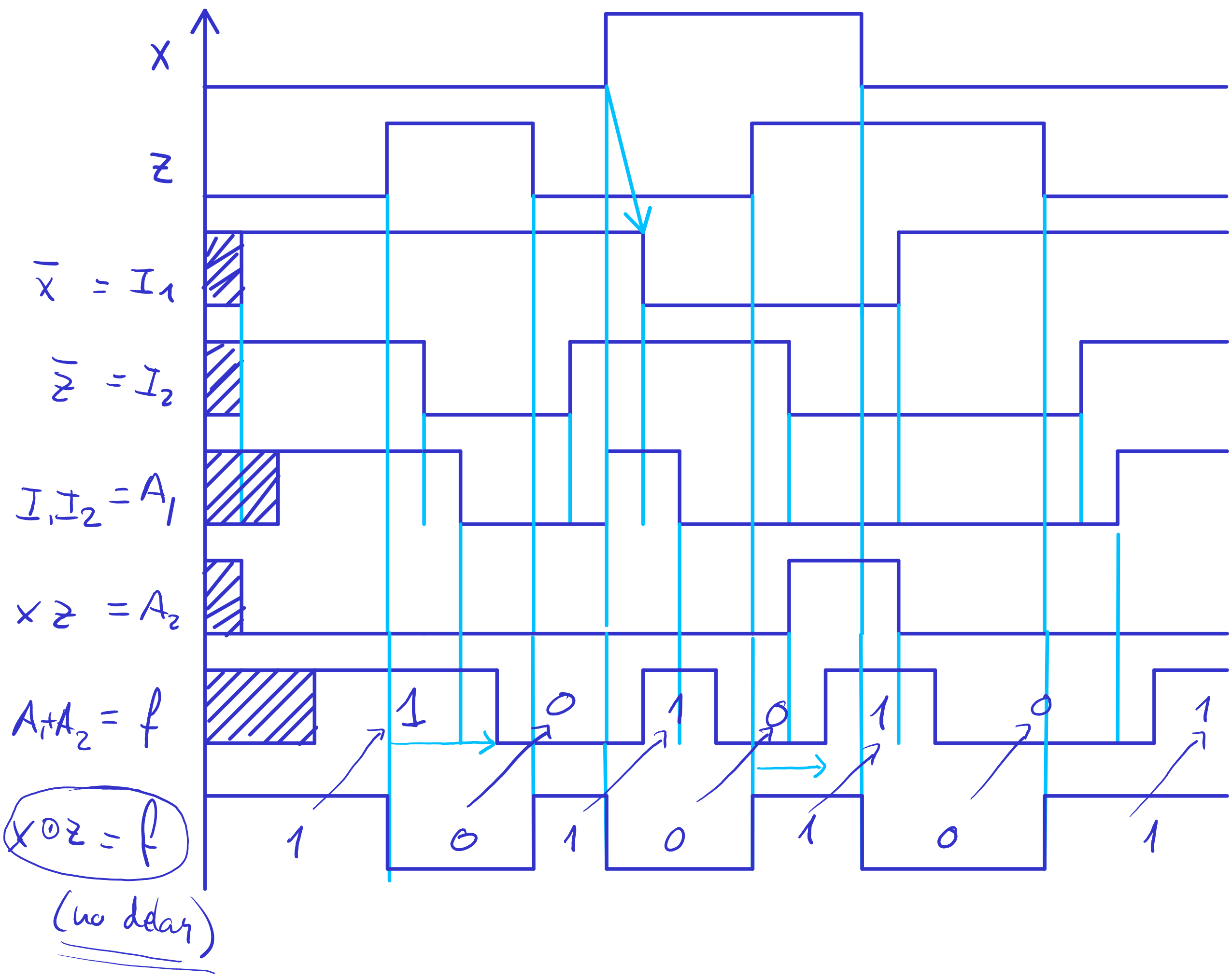
$I_2 = \bar{z}$

$A_1 = I_1 I_2$

$A_2 = xzy \rightarrow A_2 = xz$

$f = A_1 + A_2$

$$f = \bar{x}\bar{z} + xyz \qquad f(y=1) = \bar{x}\bar{z} + xz = x \odot z = \overline{x \oplus z}$$

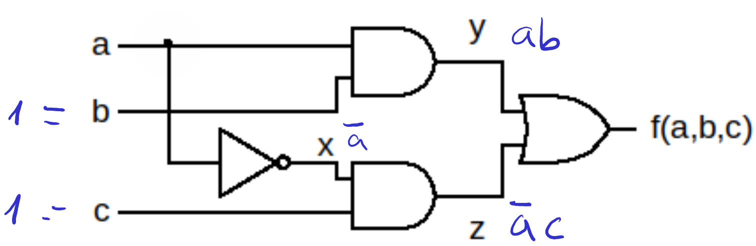


Example 15. Given the circuit in the picture:

a) Obtain the chronogram of "f" considering that all gates have the same delay, b=c=1 and that "a" changes periodically.

b) ¿Does the output present any hazards? If yes, re-desing the circuit to avoid the possibility of output hazards.

c) Obtain the chronogram of the new circuit to check.



$b=1,\ c=1$

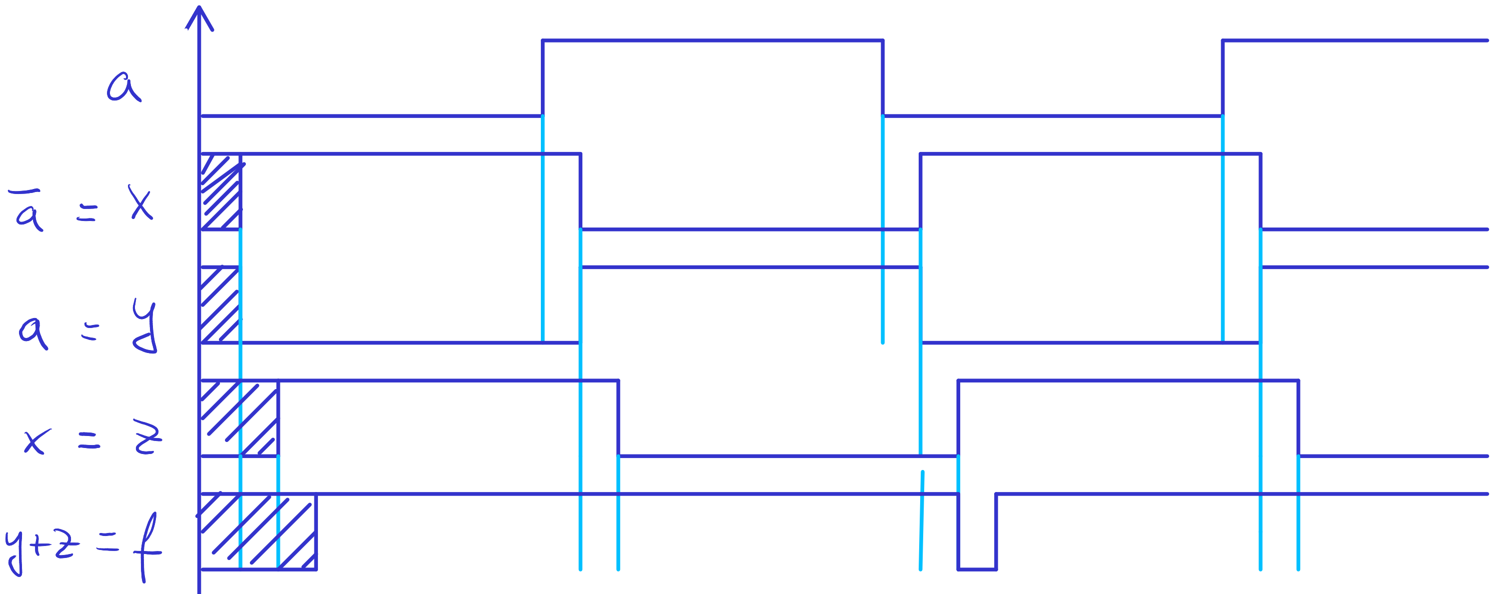
$y = ab$

$x = \bar{a}$

$z = xc$

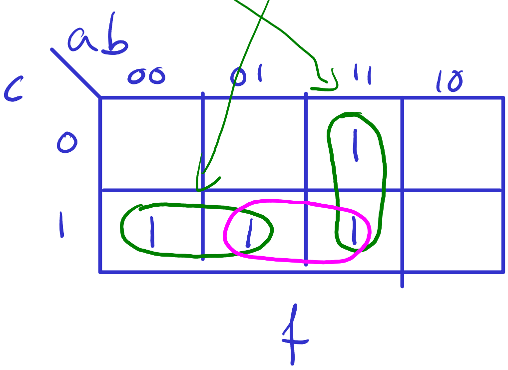
$f = y + z$

$$\boxed{\begin{array}{l} y = a \\ x = \bar{a} \\ z = x \\ f = y + z \end{array}}$$



$f = a + x = a + \bar{a} = 1$

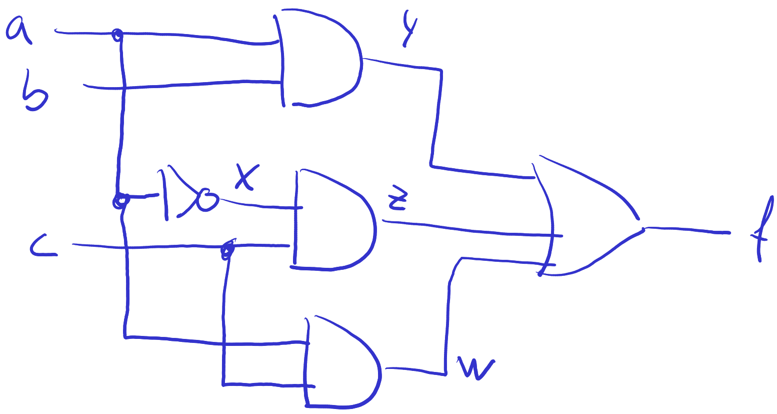
$$f = ab + \bar{a}c$$



$$f = ab + \bar{a}c + bc$$

$$\boxed{b = c = 1}$$

$$f(b=c=1) = a + \bar{a} + 1 = 1$$



$$y = ab = a$$

$$x = \bar{a}$$

$$z = xc = x$$

$$w = bc = \underline{\underline{1}}$$

$$\underbrace{f = y + z + w = 1}$$

We can see that $f = 1$ no matter what happens with y or z => there will be no hazards, but we can draw the chronogram to make it more clear